3. Kähler manifolds

A Kähler manifold is a complex manifold $M$ with a Kähler form $\omega$ which is closed $(d\omega = 0)$. A Kähler form is equivalent to a Hermitian metric $h$. We define these and show how they are related on a single vector space $V$, then on the tangent bundle of $M$. (However, on a single vector space, it doesn’t make sense to talk about closed forms.)

3.1. Kähler forms. We started with the basic concept of a Kähler form. Suppose that $V$ is a vector space over $\mathbb{C}$: $V \cong \mathbb{C}^n$ and $W_{\mathbb{R}} = \text{Hom}(V, \mathbb{R})$,

$$W_{\mathbb{C}} = W_{\mathbb{R}} \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}.$$

Recall that $W_{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ and $W^{1,0} = \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) \subset W_{\mathbb{C}}$. Now take $W^{1,1} = W^{1,0} \otimes W^{0,1}$. We want to look at

$$W^{1,1}_{\mathbb{R}} = W^{1,1} \cap \wedge^2 W_{\mathbb{R}}.$$

Thus elements of $W^{1,1}_{\mathbb{R}}$ are alternating real forms of type $(1,1)$.

Example 3.1.1. The basic example is $V = \mathbb{C}^n$,

$$\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j = \frac{i}{2} \sum (dx_j + i dy_j) \wedge (dx_j - i dy_j) = \sum dx_j \wedge dy_j.$$

Since $a \wedge b = a \otimes b - b \otimes a$, for $z = x + iy, z' = x' + iy' \in \mathbb{C}^n$ this form is

$$\omega(z, z') = \sum_j (x_j y'_j - y_j x'_j).$$

The first form of $\omega$ shows that it lies in $W^{1,1}$. The last form shows it is in $\wedge^2 W_{\mathbb{R}}$. The scalar $\frac{i}{2}$ is needed to make the form real. All Kähler forms will be equivalent to these.

Lemma 3.1.2. $\omega \in W^{1,1}_{\mathbb{R}}$ if and only if $\omega : V \times V \to \mathbb{R}$ is a skew symmetric $\mathbb{R}$-bilinear form so that

(3.1) \hspace{1cm} \omega(Iu, Iv) = \omega(u, v)

for all $u, v \in V$.

Proof. First note that the condition $\omega(Iu, Iv) = \omega(u, v)$ is equivalent to the condition

(3.2) \hspace{1cm} \omega(u, Iv) + \omega(Iu, v) = 0

since $I^2 = -1$. By definition, $W^{1,1}_{\mathbb{R}}$ is the set of skew-symmetric forms $\omega$ on $V$ so that $\omega_{\mathbb{C}} = \omega \otimes \mathbb{C}$ lies in $W^{1,1}$. This is equivalent to the condition that $\omega_{\mathbb{C}}$ vanishes on pairs of vectors from $V^{1,0}$ or from $V^{0,1}$. But $V^{1,0}$ is the set of all vectors of the form

$$\tilde{u} = u - iIu$$

where $u \in V$ and

$$\omega(\tilde{u}, \tilde{v}) = \omega(u - iIu, v - iIv)$$

$$= \omega(u, v) - \omega(Iu, Iv) - i(\omega(u, Iv) + \omega(Iu, v))$$

which is zero if and only if (3.1) and (3.2) hold.
Note that (3.2) implies that

\[ g(u, v) := \omega(u, Iv) \]

is a symmetric bilinear pairing \( g : V \times V \to \mathbb{R} \) since

\[ g(v, u) = \omega(v, Iv) = -\omega(Iu, v) = \omega(u, Iv) = g(u, v) \]

**Definition 3.1.3.** A hermitian form on a complex vector space \( V \) is defined to be a map \( h : V \times V \to \mathbb{C} \) so that

1. \( h(u, v) \) is \( \mathbb{C} \)-linear in \( u \)
2. \( h(u, v) \) is \( \mathbb{C} \)-antilinear in \( v \)
3. \( h(v, u) = \overline{h(u, v)} \)

Note that (3) implies that \( h(v, v) \in \mathbb{R} \). The form \( h \) is said to be **positive definite** if \( h(v, v) > 0 \) for all \( v \neq 0 \). A positive definite hermitian form on \( V \) is also called a Hermitian metric on \( V \).

**Proposition 3.1.4.** There is a \( 1 \times 1 \) correspondence between hermitian forms \( h \) on \( V \) and forms \( \omega \in W^{1,1}_\mathbb{R} \) given by

\[ \omega = -\Im h. \]

Furthermore,

\[ h(u, v) = g(u, v) - i\omega(u, v) \]

where \( g : V \times V \to \mathbb{R} \) is given by \( g(u, v) = \omega(u, Iv) \).

**Proof.** For the second part, \( h(u, Iv) = g(u, Iv) - i\omega(u, Iv) \). Since \( h(u, v) \) is conjugate linear in \( v \), \( h(u, Iv) = -ih(u, v) = -i\omega(u, v) - ig(u, v) \). Comparing complex parts gives

\[ g(u, v) = \omega(u, Iv). \]

\[ \square \]

**Example 3.1.5.** The standard positive definite hermitian form on \( \mathbb{C}^n \) is:

\[ h(z, z') = \sum z_j \overline{z_j'} = \sum (x_j + iy_j)(x_j' + iy_j') \]

\[ = \sum \begin{cases} (x_j x_j' + y_j y_j') & g(z, z') \\ -i \sum (x_j y_j' - y_j x_j') & \omega(z, z') \end{cases} \]

**Definition 3.1.6.** A Kähler form on \( V \) is a form \( \omega \in W^{1,1}_\mathbb{R} \) whose corresponding hermitian form \( h \) is positive definite. In particular, Kähler forms are nondegenerate.

3.2. **Kähler metrics.** Suppose that \( (M, I) \) is an almost complex manifold. Then a Hermitian metric \( h \) on \( M \) is a Hermitian metric \( h_x \) on the tangent space \( T_{M,x} \) at each point which varies smoothly with \( x \in M \). Associated to \( h \) we have:

\[ \omega = -\Im h \]

which is a 2-form on \( M \) which is also in \( \Omega^{1,1}_M \) which is equivalent to the equation

\[ \omega(Iu, Iv) = \omega(u, v) \]

for any two vectors \( u, v \in T_{M,x} \) at any point \( x \in M \). By Definition 3.1.6, \( \omega \) is a Kähler form. However we usually want \( \omega \) to be closed. We say that the Hermitian metric \( h \) is a Kähler metric if the corresponding Kähler form \( \omega \) is closed.
Proposition 3.2.1. The real part of a Hermitian metric $h$ is a Riemannian metric $g$ on $M$ which is also invariant under $I$:

$$g(u, v) = \omega(u, Iv) = g(Iu, Iv)$$

Proof. We know that $g$ is a symmetric real form. If $v \neq 0 \in T_{M,x}$ is a nonzero vector, $h(v, v) = \overline{h(v, v)}$ is a positive real number. So,

$$g(v, v) = h(v, v) > 0$$

So, $g$ is a Riemannian metric on $M$. \qed

Note that $M$ is oriented since any complex vector space has a natural real orientation.

Theorem 3.2.2. Given a Hermitian metric $h$ on the complex manifold $M$, the volume form on $M$ associated to the Riemannian metric $g = \Re h$ is equal to $\omega^n/n!$.

To prove this, we need the matrices for general $h \hookrightarrow \omega$ in local coordinates:

$$z = (z_1, \cdots, z_n) : U \hookrightarrow \mathbb{C}^n, \quad z_j = x_j + iy_j$$

and $\overline{z} = (\overline{z_1}, \cdots, \overline{z_n})$ centered at $x_0 \in U$. Then $dz_j, d\overline{z}_j$ form bases for $\Omega^1_{M,x_0}, \Omega^0_{M,x_0}$.

These are $W^{1,0}, W^{0,1}$ for $V = T_{M,x_0}$. So $h \in \Omega^1_M$ is given by

$$h = \sum \alpha_{ij} dz_i \otimes d\overline{z}_j$$

where $\alpha_{ij} : M \to \mathbb{C}$ (notation: $\alpha_{ij} \in \Omega^0_{M,\mathbb{C}}$). These functions are given by:

$$\alpha_{ij} = h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

Since $h(u, v) = \overline{h(v, u)}$, $\alpha_{ji} = \overline{\alpha_{ij}}$. Since $-3z = \frac{i}{2}(z - \overline{z})$, we have:

$$\omega = -3h = \frac{i}{2} \sum (\alpha_{ij} dz_i \otimes d\overline{z}_j - \alpha_{ji} d\overline{z}_i \otimes dz_j) = \frac{i}{2} \sum \alpha_{ij} dz_i \wedge d\overline{z}_j.$$

Proof of Theorem 3.2.2. By a $\mathbb{C}$-linear change of the coordinates $z = (z_1, \cdots, z_n)$, we can arrange for $\frac{\partial}{\partial \overline{z}_i}$ to be ortho-normal at the point $x_0$. In other words,

$$\alpha_{ij}(x_0) = h_{x_0} \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \delta_{ij}$$

Then, at $x_0$,

$$\omega_{x_0} = \frac{i}{2} \sum dz_j \wedge d\overline{z}_j = \sum dx_j \wedge dy_j$$

by Example 3.1.1. So,

$$\omega^n = \sum_{\pi \in S_n} dx_{\pi(1)} \wedge dy_{\pi(1)} \wedge dx_{\pi(2)} \wedge dy_{\pi(2)} \wedge \cdots \wedge dx_{\pi(n)} \wedge dy_{\pi(n)}$$

which is $n!$ times the volume form $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ at the point $x_0$. Since this hold at every point $x_0 \in M$, $\omega^n/n!$ is the volume form at each point. \qed
3.3. Kähler manifolds.

**Definition 3.3.1.** A Kähler manifold is a complex manifold with a Kähler metric which is, by definition, a Hermitian metric $h$ so that $\omega$ is closed ($d\omega = 0$).

**Corollary 3.3.2.** On a compact Kähler manifold $M$, for every $1 \leq k \leq n$, the form $\omega^k$ is closed but not exact. I.e., $[\omega^k] \neq 0 \in H^{2k}(M)$.

*Proof.* $\omega^k$ is clearly closed:

$$d\omega^k = k\omega^{k-1}d\omega = 0$$

If $\omega^k = d\alpha$ then

$$d(\alpha \wedge \omega^{n-k}) = d\alpha \wedge \omega^{n-k} = \omega^n$$

which is impossible since the volume form is nonzero in $H^{2n}(M)$ when $M$ is an oriented compact manifold. \qed

**Corollary 3.3.3.** A compact complex $k$-submanifold $N$ of a Kähler manifold $M$ cannot be the boundary of a (real) submanifold of $M$.

*Proof.* This follows immediately from Stokes' Theorem. If $N = \partial W$ then:

$$\int_W d\omega^k = 0 = \int_N \omega^k = \text{vol } N$$

a contradiction, since $\omega^k$ is the volume form on $N$. \qed

3.4. Connections. Given a real $C^\infty$ $k$-dimensional vector bundle $E$ on a real manifold $X$ we want to take the derivative of a section of $E$. This is given by a connection. Recall that a connection $\nabla$ on $E$ is a linear map

$$\nabla : A^0(E) \to A^1(E) = \Gamma(T^*_X \otimes_R E)$$

satisfying the Leibnitz equation

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla \sigma$$

for all $f : X \to \mathbb{R}$ ($f \in \mathcal{O}_X^0$), $\sigma \in \Gamma E = A^0(E) = C^\infty(E)$.

Thus, $\nabla$ takes a section $\sigma$ of $E$ and gives $\nabla(\sigma) \in A^1(E)$ which is a 1-form on $X$ with coefficients in $E$:

$$\nabla(\sigma) \in A^1(E) = \Gamma(T^*_X \otimes_R E) = \Gamma \text{Hom}_\mathbb{R}(T_X, E).$$

We interpret $\nabla(\sigma)$ as the derivative of $\sigma$.

Analog: Let $f : X \to \mathbb{R}$ be a smooth function. Then $D(f) = df : T_X \to \mathbb{R}$ is a 1-form on $X$:

$$df \in \Gamma T^*_X = \Gamma \text{Hom}(T_X, \mathbb{R}).$$

Given a vector field $\psi$ on $X$, $D_\psi(f) = df(\psi)$ is the directional derivative of $f$ in the direction $\psi$. This is at every $x \in X$. So, $D_\psi(f) \in \mathcal{O}_X^0$ is a smooth function on $X$.

For a connection we write:

$$\nabla_\psi(\sigma) := \nabla(\sigma)\psi \in \Gamma E.$$

This is the $\nabla$-directional derivative of the section $\sigma$ in the direction of the tangent vector field $\psi$. 
Following our philosophy of concentrating on concepts and definitions (and skipping proofs of theorems), we reformulate the definition of a connection.

**Lemma 3.4.1.** If $E, E'$ are vector bundles on $X$ then

$$\text{Hom}_X(E, E') = \Gamma \text{Hom}_\mathbb{R}(E, E') = \text{Hom}_{\Omega^0_X}(\Gamma E, \Gamma E').$$

**Proof.** Locally, a homomorphism of vector bundles $E^k \to E'^\ell$ is given by a map

$$M : X \to \text{Hom}_\mathbb{R}(\mathbb{R}^k, \mathbb{R}^\ell)$$

which is a family of $\ell \times k$ matrices $M(x)$, with entries $m_{ij} : X \to \mathbb{R}$.

Locally, sections of $E = X \times \mathbb{R}^k$ are given by $k$ functions on $X$:

$$u \in \Gamma E = (u_1, \ldots, u_k) = \sum u_i e_i$$

where $e_i$ are “basic sections” of $E$ and $u_i \in \Omega^0_X$. I.e., $\Gamma E$ is a free $\Omega^0_X$-module. An $\Omega^0_X$-morphism $M : \Gamma E \to \Gamma E'$ is therefore given by:

$$M(\sigma) = \sum u_i M(e_i) = \sum u_i m_{ij} e_j' = (u_1, \ldots, u_k)M$$

So, $M$ is given by the same data: a matrix with entries $m_{ij} \in \Omega^0_X$. $\square$

For a connection $\nabla : \Gamma E \to A^1(E)$, $\Gamma E$, $A^1(E)$ are both $\Omega^0_X$ modules. But $\nabla$ is not a homomorphism of $\Omega^0_X$ modules:

$$\nabla(f \sigma) = df \sigma + f \nabla(\sigma).$$

However, if we have another connection $\nabla'$,

$$\nabla'(f \sigma) = df \sigma + f \nabla'(\sigma).$$

So, the difference

$$(\nabla - \nabla')(f \sigma) = f(\nabla - \nabla')\sigma$$

is a homomorphism of $\Omega_X$-modules. Analogously to the proof of Lemma 3.4.1 such morphisms are given by matrices $M = (m_{ij})$ with $m_{ij} \in \Omega^1_X$.

In local coordinates, $d$ is a connection. So, an arbitrary connection is given by $\nabla = d + \varphi$ or:

$$\nabla(f_1, \ldots, f_k) = (df_1, \ldots, df_k) + (f_1, \ldots, f_k)M$$

where $M$ is a $k \times k$ matrix with entries in $\Omega^1_X$.

If $X$ has a Riemannian metric $g$ then recall that the **Levi-Civita connection** $\nabla = \nabla^{LC}$ is the unique connection on $E = T_M$ having the properties:

1. $dg(\sigma, \tau) = g(\nabla \sigma, \tau) + g(\sigma, \nabla \tau)$ (\nabla is compatible with $g$)
2. $\nabla_\sigma(\tau) - \nabla_\tau(\sigma) = [\sigma, \tau]$ usually written as:

$$\nabla_\sigma(\tau) - \nabla_\tau(\sigma) = [\sigma, \tau]$$

for any two vector fields $\sigma, \tau$ on $X$.

As discussed in class, Equation (1) says that, for any vector field $\psi$ on $X$,

$$dg(\sigma, \tau)(\psi) = g(\nabla_\psi \sigma, \tau) + g(\sigma, \nabla_\psi \tau).$$
Explanations: Since \( g(\sigma, \tau) \in \Omega^1_X \), all three terms in Equation (1) are 1-forms on \( X \). For example, if \( \nabla \sigma = \sum \xi_i \alpha_i \) where \( \xi_i \) are vector fields and \( \alpha_i \) are 1-forms on \( X \), then
\[
g(\nabla \sigma, \tau) = \sum g(\xi_i, \tau) \alpha_i.
\]
Applying both sides to the vector field \( \psi \) we get:
\[
g(\nabla \sigma, \tau)(\psi) = \sum g(\xi_i, \tau) \alpha_i(\psi) = \sum g(\xi_i \alpha_i(\psi), \tau) = g(\nabla \sigma(\psi), \tau) = g(\nabla_\psi \sigma, \tau).
\]
Recall that a connection on a smooth bundle \( E \) over \( X \) is a linear map
\[
\nabla : \Gamma E = A^0(E) \to A^1(E)
\]
satisfying Leibniz rule. When \( X \) is a complex manifold and \( E \) is a holomorphic bundle, \( A^0(E), A^1(E) \) are the same set as before but with more structure:
\[
A^1(E) = \Gamma(T_X^* \otimes_R E) = \Gamma(T_X^* \otimes_C E) = A^{1,0}(E) \otimes A^{0,1}(E)
\]
and \( A^0(E) = A^{0,0}(E) \) is still the space of smooth sections of \( E \). Then, any connection
\[
\nabla : A^{0,0}(E) \to A^1(E) = A^{1,0}(E) \otimes A^{0,1}(E)
\]
has two components \( \nabla^{1,0}, \nabla^{0,1} \). Last time we showed that
\[
\bar{\partial}_E : A^{0,0}(E) \to A^{0,1}(E)
\]
given in local coordinates by \( \bar{\partial}_E(f_1, \cdots, f_k) = (\bar{\partial}f_1, \cdots, \bar{\partial}f_k) \) is well defined.

**Proposition 3.4.2.** Given a Hermitian metric \( h \) on a holomorphic bundle \( E \), there is a unique connection \( \nabla \) on \( E \) so that
1. \( dh(\sigma, \tau) = h(\nabla \sigma, \tau) + h(\sigma, \nabla \tau) \) for all \( \sigma, \tau \in A^{0,0}(E) \) (\( \nabla \) is “compatible” with \( h \))
2. \( \nabla^{0,1} = \bar{\partial}_E \)

This unique connection is called the Chern connection on \( E \).

**Proof.** \( \nabla = \nabla^{1,0} + \nabla^{0,1} \) where \( \nabla^{0,1} = \bar{\partial}_E \) and \( \nabla^{1,0} \) is uniquely determined by:
\[
(3.3) \quad dh(\sigma, \tau) = h(\nabla^{1,0} \sigma, \tau) + h(\sigma, \bar{\partial} \tau)
\]
since \( h(\Delta^{0,1} \sigma, \tau) = 0 \) and \( h(\sigma, \Delta^{1,0} \tau) = 0 \). In more detail, let \( \nabla^{1,0} = \partial + M \) where, in local coordinates, \( M \) is a \( k \times k \) matrix with entries in \( \Omega^1_X \). We have:
\[
h(u, v) = \sum h_{ij} u_i v_j
\]
where
\[
h_{ij} = h \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right).
\]
Then
\[
dh(u, v) = h(\partial u, v) + h(u, \bar{\partial} v) + \sum dh_{ij} u_i v_j
\]
and
\[
h(\nabla^{1,0} u, v) = h(\partial u, v) + h(u M, v).
\]
Let \( M \) be the solution of the equation
\[
h(u M, v) = \sum dh_{ij} u_i v_j
\]
which is bilinear in \( u, v \). Then \( \nabla^{1,0} = \partial + M \) will satisfy (3.3). \( \square \)
Theorem 3.4.3. Let \((M, I)\) be a complex manifold with a Hermitian metric \(h = g - \omega\) and let \(\nabla^{LC}\) be the Levi-Civita connection for \(g\). Then the following are equivalent.

1. \(h\) is a Kähler metric \((d\omega = 0)\).
2. \(\nabla^{LC}(I\sigma) = I\nabla^{LC}(\sigma)\) for every real vector field \(\sigma\) on \(X\).
3. The holomorphic part of the Chern connection is equal to the Levi-Civita connection:

\[
\mathcal{R}(\nabla^{1,0}) = \nabla^{LC}
\]

(See Fig 1.) So, the Chern connection on \(T_{X,\mathbb{C}}\) is \((\nabla^{LC}, \overline{\partial})\).

**Proof.** (3) \(\Rightarrow\) (2). The Chern connection is complex linear and the holomorphic part is the part where \(i = I\):

\[
\mathcal{R}(i\sigma) = I\mathcal{R}(\sigma)
\]

for \(\sigma\) a section of \(T^{1,0}_X\). So,

\[
\nabla^{LC}(I\sigma) = \mathcal{R}(\nabla^{1,0}(I\sigma)) = \mathcal{R}(i\nabla^{1,0}(\sigma)) = I\mathcal{R}(\nabla^{1,0}(\sigma)) = I\nabla^{LC}(\sigma).
\]

(2) \(\Rightarrow\) (1). Since \(\nabla = \nabla^{LC}\) is compatible with \(g = \mathcal{R}(h)\), we are given that

\[
d(g(\sigma, \tau)) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau).
\]

Replacing \(\tau\) with \(I\tau\) we get \(g(\sigma, I\tau) = \omega(\sigma, \tau)\) and \(\nabla I\tau = I\nabla\tau\) by (2). So,

\[
d(\omega(\sigma, \tau)) = \omega(\nabla\sigma, \tau) + \omega(\sigma, \nabla\tau).
\]

Apply both to vector field \(\phi\), use rule \(df(\phi) = \phi(f)\) and skew-symmetry of \(\omega\):

\[
\phi(\omega(\sigma, \tau)) = \omega(\nabla_{\phi}\sigma, \tau) - \omega(\nabla_{\phi}\tau, \sigma).
\]

Cyclically permute the three vector fields:

\[
\omega(\sigma(\tau, \phi)) = \omega(\nabla_{\sigma}\tau, \phi) - \omega(\nabla_{\sigma}\phi, \tau)
\]
Proof: Since $\phi$ is a linear combination of $z_k$ ($\epsilon_{ij}$ are holomorphic) 

$$
\epsilon_{ij} = \sum_c \epsilon_{ij}^k z_k 
$$

and $\epsilon'_{ij}$ is a linear combination of $\overline{z}_k$ ($\epsilon'_{ij}$ are antiholomorphic):

$$
\epsilon'_{ij} = \sum_c \epsilon'_{ij}^k \overline{z}_k 
$$

Since $h$ is conjugate symmetric we have:

$$
\epsilon'_{ij} = \overline{\epsilon_{ji}}
$$

The key property of these numbers is:

**Claim:** If $h$ is Kähler then 

$$
\epsilon_{ij}^k = \epsilon'_{kj}^i
$$

Proof: Since $\partial \delta_{ij} = 0$ and $\partial \overline{z}_k = 0 \Rightarrow \partial \epsilon'_{ij} = 0$, at the point $z = 0$ we have:

$$
0 = \partial \omega = \frac{i}{2} \sum_{ij} \partial \epsilon_{ij} dz_i \wedge d\overline{z}_j - \frac{i}{2} \sum_{ijk} \epsilon_{ij}^k d z_k \wedge d z_i \wedge d \overline{z}_j
$$

where we recall that the $\frac{i}{2}$ factor comes from: $-\Re(z) = \frac{i}{2} (z - \overline{z})$ (and $d \omega = 0$ is equivalent to $\partial \omega = 0 = \overline{\partial} \omega$). In order for these terms to cancel, we must have $\epsilon_{ij}^k = \epsilon'_{kj}^i$ as claimed.
Now let
\[ z'_j = z_j + \frac{1}{2} \sum \epsilon_{ij}^k z_i z_k. \]
Since \( \epsilon_{ij}^k z_i z_k \) is symmetric in \( z_i, z_k \), we have
\[ dz'_j = dz_j + \sum \epsilon_{ij}^k z_k dz_i = dz_j + \sum \epsilon_{ij} dz_i = dz_j + O(|z|) \]
which implies;
\[ dz_j = dz'_j - \sum \epsilon_{ij} dz'_i + O(|z|^2) \]
\[ \frac{\partial}{\partial z'_i} = \sum \frac{\partial z_j}{\partial z'_i} \frac{\partial}{\partial z_j} = \sum (\delta_{ij} - \epsilon_{ij}) \frac{\partial}{\partial z_j} + O(|z|^2) \]
So, up to terms of second order, we have:
\[ h'_{ij} = h \left( \frac{\partial}{\partial z'_i}, \frac{\partial}{\partial z'_j} \right) = \sum \delta_{ik} - \epsilon_{ik} h_{k\ell} (\delta_{j\ell} - \epsilon_{j\ell}) \]
\[ = \sum \delta_{ik} - \epsilon_{ik} (\delta_{k\ell} + \epsilon_{k\ell} + \epsilon'_{k\ell}) (\delta_{j\ell} - \epsilon'_{j\ell}) \]
since \( \epsilon'_{j\ell} = \epsilon'_{j\ell} \),
\[ = \sum \delta_{ik} \delta_{j\ell} - \epsilon_{ik} \delta_{k\ell} \delta_{j\ell} + \delta_{ik} \epsilon_{k\ell} \delta_{j\ell} + \delta_{ik} \epsilon'_{k\ell} \delta_{j\ell} - \delta_{ik} \delta_{k\ell} \epsilon'_{j\ell} \]
\[ = \delta_{ij} - \epsilon_{ij} + \epsilon_{ij} + \epsilon'_{ij} - \epsilon'_{ij} = \delta_{ij} \]
In other words, the matrix \( (h'_{ij}) \) of \( h \) with respect to the new coordinates \( z'_j \) is equal to the identity matrix up to second order. This proves the Lemma and completes the proof of Theorem 3.4.3. \( \square \)
3.5. Examples of Kähler manifolds. An easy example is a Riemann surface. This is a complex 1-dimensional and real 2-dimensional manifold. Any Hermitian metric is Kähler since all 2-forms on a real 2-dimensional manifold are closed.

The next example is $\mathbb{C}P^n = \mathbb{P}^n(\mathbb{C})$. We will construct the Fubini-Study metric on complex projective space $\mathbb{P}^n(\mathbb{C})$ and showed that it is a Kähler metric. This will imply that all smooth projective varieties over $\mathbb{C}$ are Kähler manifolds.

The outline of the construction is:

$$L \mapsto (L, h) \mapsto \omega_L \leftrightarrow h_{\omega}$$

Given a holomorphic line bundle $L$ on a complex manifold $X$, chose a hermitian form $h$ on $L$. Then, there is an associated 2-form $\omega_L$ on $X$ (called the Chern form of $(L, h)$). This 2-form $\omega_L$ is associated to a Hermitian metric $h_{\omega}$ on $X$ ($h_{\omega} \neq h$) which, if we are lucky, will be positive definite and therefore a Kähler metric. We will apply this to the canonical line bundle $S^*$ over $X = \mathbb{P}^n(\mathbb{C})$ to obtain the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$.

A line bundle $L$ over $X$ is the union over open sets $U_i$ of $U_i \times \mathbb{C}$. For each $U_i$, take the unit section $\sigma_i(v) = (v, 1)$. These in general don’t match. So, there are functions $g_{ij}: U_i \cap U_j \to \mathbb{C}^\times$ so that

$$\sigma_i(v) = g_{ij}(v)\sigma_j(v)$$

for all $v \in U_i \cap U_j$. Since $\sigma_j = g_{jk}\sigma_k$ we have $\sigma_i = g_{ij}\sigma_j = g_{ij}g_{jk}\sigma_k$. So,

$$g_{ik} = g_{ij}g_{jk}$$

on $U_i \cap U_j \cap U_k$. Conversely, any collection of maps $g_{ij}: U_i \cap U_j \to \mathbb{C}^\times$ satisfying the equations in the box will uniquely determine a line bundle. If the $g_{ij}$ are holomorphic functions, the line bundle will be a holomorphic bundle.

For example, $g_{ij}' := \frac{1}{g_{ij}}$ is another collection of functions satisfying the same (boxed) equations. So, $\{g_{ij}'\}$ gives another holomorphic line bundle $L^*$ which one can show is the dual bundle to $L$.

Let $h$ be a Hermitian metric on $L$. Let $h_i: U_i \to \mathbb{R}^+$ be the positive function given by $h_i(v) = h(\sigma_i(v), \sigma_i(v))$. Since $\sigma_i = g_{ij}\sigma_j$ we get:

$$h_i = h(\sigma_i, \sigma_i) = g_{ij}\overline{g}_{ij}h(\sigma_j, \sigma_j) = g_{ij}\overline{g}_{ij}h_j$$

Lemma 3.5.1. Conversely, any family of functions $h_i: U_i \to \mathbb{R}^+$ satisfying the equations $h_i = g_{ij}\overline{g}_{ij}h_j$ gives a Hermitian metric on $L$.

Proof. Let $h_i'$ be another collections of functions so that $h_i' = g_{ij}\overline{g}_{ij}h_j'$. On each $U_i$ let

$$f_i = h_i'/h_i.$$  Then $f_j = h_j'/h_j = g_{ji}\overline{g}_{jj}h'_j/h_i = h_j'/h_i = f_i$. So, $f = f_i = f_j$ is a globally defined function on $X$ and $h' = fh$ is another metric on $L$.  \qed

For example, $h_i^* = \frac{1}{h_i}$ satisfies

$$h_i^* = g_{ij}\overline{g}_{ij}^* h_j^*$$

Therefore, $h_i^*$ gives a metric on $L^*$.

Let $$\omega_i = \frac{1}{2\pi i} \partial\overline{\partial} \log h_i.$$
Note that
\[ \log h_i = \log g_{ij} + \log \overline{g_{ij}} + \log h_j. \]
Since \( g_{ij} \) is holomorphic, \( \partial \log g_{ij} = 0 \). Since \( \overline{g_{ij}} \) is antiholomorphic, \( \partial \log \overline{g_{ij}} = 0 \). So,
\[ \omega_i = \frac{1}{2\pi i} \partial \overline{\partial} \log h_i = \frac{1}{2\pi i} \partial \overline{\partial} \log h_j = \omega_j. \]
So, \( \omega = \omega_i \) is a well-defined 2-form on all of \( X \). Also \( d\omega = \partial \omega + \overline{\partial} \omega = 0 \) since \( \partial^2 = 0 = \overline{\partial}^2 \).

**Theorem 3.5.2.** Given a holomorphic line bundle \( L \) on a complex manifold \( X \) and a Hermitian metric \( h \) on \( L \), there is a closed form \( \omega \) on \( X \) of type \( (1,1) \) given locally by
\[ \omega = \frac{1}{2\pi i} \partial \overline{\partial} \log h. \]
We call \( \omega \) the Chern form of \( (L,h) \).

Now let \( X = \mathbb{P}^n(\mathbb{C}) \). Recall that this is the quotient space of \( \mathbb{C}^{n+1}\setminus 0 \) modulo the relation
\[ (z_0, z_1, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n) \]
for any \( \lambda \neq 0 \in \mathbb{C} \). The equivalence class is denoted \([z_0, \ldots, z_n]\). Another interpretation is that \( \mathbb{P}^n(\mathbb{C}) \) is the set of one dimensional subspaces \( \Delta \) of \( \mathbb{C}^{n+1} \). Each such \( \Delta \) is uniquely determined by any nonzero vector \((z_0, \ldots, z_n) \in \Delta \) and we make the identification \( \Delta = [z_0, \ldots, z_n] \).

Let \( S \) be the tautological line bundle over \( \mathbb{P}^n(\mathbb{C}) \) given by
\[ S = \{ (\Delta, v) \mid \Delta \in \mathbb{P}^n(\mathbb{C}) \text{ and } v \in \Delta \} \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}. \]
This is “tautological” since the fiber over the point \( \Delta \in \mathbb{P}^n(\mathbb{C}) \) is the space \( \Delta \subset \mathbb{C}^{n+1} \).

Let \( U_i \) be the open subset of \( \mathbb{P}^n \) given by
\[ U_i = \{ [z] \mid z_i \neq 0 \}. \]
Let \( \sigma_i \) be the section of \( S \) over \( U_i \) given by
\[ \sigma_i(\Delta) = \sigma_i([z_0, \ldots, z_n]) = \left( \frac{z_0}{z_i}, \ldots, \frac{z_i}{z_i} = 1, \ldots, \frac{z_n}{z_i} \right). \]
This is well-defined since, e.g., the \( j \)th coordinate is
\[ \frac{z_j}{z_i} = \frac{\lambda z_j}{\lambda z_i}. \]
\( \sigma_i(\Delta) \) is the unique element of \( \Delta \) with \( i \)th coordinate equal to 1. Comparing this with
\[ \sigma_j([z]) = \left( \frac{z_0}{z_j}, \ldots, \frac{z_i}{z_j}, \ldots, \frac{z_n}{z_j} \right) \]
we see that
\[ \sigma_i = \frac{z_j}{z_i} \sigma_j. \]
So, the transition functions for \( S \) are \( g_{ij} = z_j/z_i \) with dual \( g^*_{ij} = z_i/z_j \).
Since the line bundle $S$ is a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ it gets a metric by restricting the standard metric on $\mathbb{C}^{n+1}$ given by $h(z, z') = \sum z_j \bar{z}_j$. So $h(z, z) = \sum |z_j|^2$. Since the $i$th coordinate of $\sigma_i$ is 1 we get:

$$h(\sigma_i) = 1 + \sum_{j \neq i} |z_j|^2.$$ 

On the dual bundle $S^\ast$ (called the canonical bundle over $\mathbb{P}^n$) we have

$$h^\ast(\sigma_i^\ast) = \frac{1}{1 + \sum |z_j|^2}.$$ 

Using $|z_j|^2 = z_j \bar{z}_j$, the Chern form of $S^\ast$ on $U_i$ is

$$\omega_i = \frac{1}{2\pi i} \partial \bar{\partial} \log \left( \frac{1}{1 + \sum z_j \bar{z}_j} \right).$$

We calculated this step by step using first the equation

$$\partial \log \frac{1}{f} = - \frac{\partial f}{f} = -\frac{1}{f} \sum \frac{\partial f}{\partial \bar{z}_j} dz_j$$

to get:

$$\partial \log \left( \frac{1}{1 + \sum z_j \bar{z}_j} \right) = -\frac{1}{1 + \sum |z_j|^2} \sum z_j dz_j.$$ 

Apply $\partial$ using the quotient rule to get:

$$\partial \bar{\partial} \log \left( \frac{1}{1 + \sum z_j \bar{z}_j} \right) = -\left( \frac{1 + \sum |z_j|^2}{1 + \sum |z_j|^2} \right) \sum d\bar{z}_j \wedge d\bar{z}_j - \sum z_i \bar{z}_j dz_i \wedge d\bar{z}_j$$

where we used the formula $\partial(f dz_j) = \sum \frac{\partial f}{\partial \bar{z}_i} dz_i \wedge d\bar{z}_j$. At the origin $z = 0$ we get

$$\omega = \frac{i}{2\pi} \sum d\bar{z}_j \wedge d\bar{z}_j$$

which is the standard form corresponding to (a positive scalar multiple of) the standard metric with matrix equal to the identity matrix (divided by $\pi$). So, the corresponding metric $h_\omega$ is positive definite at the point $z = 0$. However, the space $\mathbb{P}^n(\mathbb{C})$ is homogeneous (the same at every point). This is easier to see if we use a vector space without a basis: Let $V$ be any $n + 1$ dimensional vector space over $\mathbb{C}$ and let $\mathbb{P}(V)$ be the space of 1-dimensional subspaces of $V$. Then it is clear that every point is the same as every other point. The tautological bundle $S$ and its dual are also defined without choice of coordinates. So, we can choose coordinates to make any point the center point $z = 0$. So, the canonically defined metric $h_\omega$ is positive definite at every point.

**Theorem 3.5.3.** The hermitian form $h_\omega$ corresponding to the canonical Chern form $\omega$ on the dual $S^\ast$ of the tautological line bundle over $\mathbb{P}^n(\mathbb{C})$ is positive definite and therefore a Kähler metric.

This form $h_\omega$ is called the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$.

**Corollary 3.5.4.** Every smooth projective variety over $\mathbb{C}$ is a Kähler manifold.